# Towards Proof-Relevant Interpolation for Circular Proofs

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## 1 Introduction

The aim of this abstract is to report on an ongoing work investigating a proof-relevant and validity-preserving interpolation theorem for the linear logic with fixed points, known as $\mu\mathsf{LL}$. In particular, we focus on strongly valid, cut-free and one-sided proofs within its circular proof system, denoted $\mu\mathsf{LL}^{\omega}$ [Dou17].

While interpolation has been extensively studied in first-order logic, its formulation in the context of fixed-point logics remains an active area of research. Moreover, our approach does not focus on the traditional notion of Craig (resp. Lyndon) interpolation [Cra57] but a proof-relevant refinement recently considered by Saurin [Sau25], whose results we extend in this study.

By *proof-relevant interpolation* we mean that not only the logical entailment is factored through an interpolant formula $I$ but rather that a proof $\pi$ of $A \vdash B$ is factored into two proofs $\pi_1$ (resp. $\pi_2$) of $A \vdash I$ (resp. $I \vdash B$), somehow amounting to a cut-introduction, a remark that will be made technically precise[1]. Our work can therefore be viewed as aiming to provide a proof-relevant version of Afshari & Leigh interpolation result for the $\mu$-calculus [AL19], refined to to hold in Linear logic.

This abstract is organised as follows: (i) we first provide some background on proof-relevant interpolation in $\mathsf{LL}$, then (ii) briefly recall background on $\mu\mathsf{LL}^{\omega}$ before (iii) turning to the question of proof-relevant interpolation for circular proofs.

Three preliminary remarks are important to clarify our point of view of this work.

- Contrary to traditional studies on interpolation properties, we do not primarily aim at interpolating logical judgements but proofs. As a result the question of the completeness of the method is to be considered with respect to a class of proofs (typically cut-free proofs or, here, strongly-valid proofs) rather than a class of logical judgements.
- Proof-relevant interpolation [Sau25] tightly connects *interpolation* with *cut introduction*. Since our present focus is on circular proofs, that means we will be confronted with what could be an infinitary cut-introduction sequence.
- While, in the realm of Craig's interpolation, the interpolant must maintain the validity (or provability) of the interpolating judgements, the shift to proof-relevant viewpoint will require synthesizing *correct* interpolating proofs that is, since we consider the case of non-wellfounded proofs, the progress condition must be ensured for both interpolating proofs.

[1] Saurin's result are connected with Čubrić results which established a proof-relevant interpolation result in natural deduction for intuitionistic logic.

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## 2 Proof-relevant interpolation in LL

In order to set the background, we briefly recall the second author's results [Sau25], proving a proof-relevant interpolation theorem for linear logic (LL) refining Maehara's method [Mae60].

In the following, we consider a one-sided sequent calculus, sequents being ordered lists of formulas and the usual cut-reduction relation written $\rightarrow_{(\mathsf{Cut})}$. The language of a formula $F$, $\mathscr{L}(F)$, consists in the set of atoms occurring in $F$.

This result can be stated as follows:

**Theorem 1.** Let $\Gamma, \Delta$ be lists of LL formulas and $\pi \vdash \Gamma, \Delta$ a cut-free proof. There exists a LL formula $I$ such that $\mathscr{L}(I) \subseteq \mathscr{L}(\Gamma) \cap \mathscr{L}(\Delta)$ and two cut-free proofs $\pi_1, \pi_2$ of $\vdash \Gamma, I$ and $\vdash I^\perp, \Delta$ respectively such that $\dfrac{\dfrac{\pi_1}{\vdash \Gamma, I} \quad \dfrac{\pi_2}{\vdash I^\perp, \Delta}}{\vdash \Gamma, \Delta}\,(\mathsf{Cut}) \rightarrow^*_{(\mathsf{Cut})} \pi$.

The proof is in two phases and amounts to a cut introduction process, synthesizing the interpolant:

**Ascending Phase:** The first phase consists of traversing the proof $\pi$ from the conclusion of the proof until the axioms, while dividing each of the sequents $\vdash \Gamma$ into a splitting $(\Gamma_l, \Gamma_r)$ inherited from the initial splitting of the conclusion, and the ancestor relation. In the end, each one of the logical axiom rules $\dfrac{}{\vdash A, A^\perp}\,(\mathsf{Ax})$ will be in one of the four following splittings: $(\{A, A^\perp\}, \{\})$, $(\{A\}, \{A^\perp\})$, $(\{A^\perp\}, \{A\})$, $(\{\}, \{A, A^\perp\})$. This also applies to the leaves of the proof trees obtained with $\dfrac{}{\vdash 1}\,(1)$ and $\dfrac{}{\vdash \top, \Gamma}\,(\top)$. Once every sequent in the proof has been split, the descending phase starts.

**Descending Phase:** Equipped with the splitting of each sequent, the cut introduction starts in the leaves of the proof and asynchronously descends to the rest of the sequents, until ultimately reaching the conclusion of the proof. An *active* sequent is a sequent where all its premises conclude with cut rules. Since $\pi$ is cut-free initially, only the leaves of the proof are trivially active (having no premise). We then apply cut introduction to active sequents, maintaining the following invariants:
(i) When a sequent is active with splitting $(\Gamma_l, \Gamma_r)$, the cut formulas of its premises are interpolants for the premise sequents with respect to their splitting.
(ii) When an inference $r$ has conclusion $c$ which is active, we apply a (sequence of) cut-introduction step(s) on this inference, in such a way that $c$ becomes the conclusion of the introduced cut and the premises of this cut correspond to the splitting associated with sequent $c$.

A similar theorem can be proved for LK and LJ sequent calculi by following the same method or by using the well-known linear embeddings of classical and intuitionistic logics.

## 3 Background on $\mu\mathsf{LL}^\omega$

We start by defining $\mu\mathsf{LL}^\omega$ formulas by extending the usual grammar of formulas $F$ in LL with 3 constructs ($F ::= \cdots \mid X \mid \mu X.F \mid \nu X.F$) and the involution on its formulas as $(X)^\perp = X$ and $(\mu X.F)^\perp = \nu X.F^\perp$ where $X$ is a *fixed-point variable* and the *least* and *greatest fixed-point operators*, denoted as $\mu$ and $\nu$ respectively, are binders. Inference rules of $\mu\mathsf{LL}^\omega$ correspond to the usual LL rules, extended with the following two fixed-point unfolding rules:

$$\dfrac{\vdash F[\nu X.F/X], \Gamma}{\vdash \nu X.F, \Gamma}\,(\nu) \qquad \dfrac{\vdash F[\mu X.F/X], \Gamma}{\vdash \mu X.F, \Gamma}\,(\mu)$$

Due to the recursive nature of the fixed-point rules, it is possible to construct infinite derivation trees even in the absence of cuts and contractions. Such infinite derivation trees are referred to as $\mu\mathsf{LL}^\infty$ *pre-proofs*. In some cases, these infinite trees contain only finitely many distinct subtrees. When this occurs, we call the derivation trees *regular*. Such regular infinite trees can be finitely represented as special graphs having the form of trees with *back-edges* (that form the cycles), yielding what are known as $\mu\mathsf{LL}^\omega$ *pre-proofs*. These proof structures are referred to as *circular pre-proofs*. The source of a back-edge is called a *bud* while its target is its *companion*. Each bud and its companion are labeled with the same sequent.

In such settings, the natural notion of subformula is the so-called Fischer-Ladner subformula relation (written $\to$) defined as follows:

$$(F \star G) \to F \qquad (F \star G) \to G \qquad (\Delta F) \to F \qquad (\sigma X.F) \to F[\sigma X.F/X]$$

where $\star \in \{⅋, \&, \otimes, \oplus\}$, $\Delta \in \{?, !, \exists_x, \forall_x\}$, and $\sigma \in \{\mu, \nu\}$.

To ensure that $\mu\mathsf{LL}^\omega$ pre-proofs represent sound arguments, they must satisfy a correctness criterion known as the *validity* or *progress* or *global trace condition*. Intuitively, a pre-proof $\mathscr{P}$ is valid if every infinite *path* in the graph induced by $\mathscr{P}$ is accompanied by an infinitely progressing *thread*. A thread is defined as a sequence of formulas $(F_i)_{i\in\omega}$ associated with a path $(v_i)_{i\in\omega}$, given by a sequence of vertices in the proof graph, such that for each index $i$, we have (i) $F_i \in s(v_i)$, (ii) $F_{i+1} \in s(v_{i+1})$ and (iii) $F_{i+1}$ is a ancestor occurrence of $F_i$ (in particular, $F_i \to F_{i+1}$ or $F_i = F_{i+1}$). Here, $s(v_k)$ refers to the sequent labelling vertex $v_k$. We say that an infinite path in $\mathscr{P}$ is valid if it admits a thread $\tau = (F_i)$ such that the minimal formula in the set of infinitely recurring formulas, denoted $\mathsf{min}(\mathsf{Inf}(\tau))$, is a $\nu$-formula, i.e. of the form $F = \nu X.F'$, where $F$ is minimal with respect to the usual subformula ordering.

In addition, we consider a stronger notion of validity, *strong validity*. An infinite path $(v_i)_{i\in\omega}$ is said to be *strongly valid* if there exists a valid thread $\tau = (F_i)_{i\in\omega}$ over $(v_i)_{i\in\omega}$, and there exists an index $k$ such that for all $h, i \geq k$, whenever $v_h = v_i$, it follows that $F_h = F_i$ (as formula occurrences of the considered sequent, *i.e.* at the same position of the list). In other words, beyond some point, the thread consistently associates the same formula with each repeated vertex.

With this in mind, it is then possible to define a $\mu\mathsf{LL}^\omega$ *strongly valid proof* as follows:

**Definition 1.** A $\mu\mathsf{LL}^\omega$ pre-proof is a *strongly valid proof*, if all of its infinite paths are strongly valid.

An essential property of strongly valid proofs is that one can extract (co)inductive invariants from the cycles and they can thus be finitized in a finitary proof system with (co)induction rules à la Park [Dou17].

# 4 Interpolating $\mu\mathsf{LL}^\omega$ circular proofs

Formally, the theorem we want to prove is the following:

**Theorem 2** (Proof-relevant Interpolation Theorem)**.** Let $\mathscr{P} \in \mu\mathsf{LL}^\omega$ be a cut-free and strongly valid proof of $\vdash \Gamma$, and $s$ a splitting of $\Gamma$ into two disjoint subsets $\Gamma_l$ and $\Gamma_r$. Then there exists a $\mu\mathsf{LL}^\omega$ formula $I$ built on the common language of $\Gamma_l$ and $\Gamma_r$ (that is $\mathscr{L}(I) \subseteq \mathscr{L}(\Gamma_l) \cap \mathscr{L}(\Gamma_r)$) and two cut-free, strongly valid proofs in $\mu\mathsf{LL}^\omega$ $\mathscr{P}_1 \vdash \Gamma_l, I$, and $\mathscr{P}_2 \vdash I^\perp, \Gamma_r$, such that $\dfrac{\dfrac{\mathscr{P}_1}{\vdash \Gamma_l, I} \quad \dfrac{\mathscr{P}_2}{\vdash I^\perp, \Gamma_r}}{\vdash \Gamma}\,(\mathsf{Cut}) \to^{\omega}_{(\mathsf{Cut})} \mathscr{P}$.

## 4.1 Issues for the proof-relevant interpolation of circular proofs

Several difficulties occur when moving from $\mathsf{LL}$ to circular proofs; they are threefold:

- The method described in Section 2 strongly relies on the ability to start the cut-introduction process from the logical axioms or unit rules. This will not be the case with circular proofs due to infinite branches or cycles: how to initiate *a(n infinitary) cut-introduction sequence*?
- Regarding cut-elimination and non-wellfounded proofs, let us stress that a cut-reduction sequence from a circular proof may have, as a limit, a cut-free non-circular proof. As a result, we will only focus on *interpolating circular cut-free proofs* (crucial to synthesize a *finite* interpolant formula).
- Finally, the interpolation process must generate two progressing proofs meaning that the interpolant has to be carefully chosen to provide enough progressing threads *via cut-introduction*. To achieve this, we will consider an additional restriction, corresponding to asking the interpolated proof to satisfy the additional *strong validity* condition considered above.

### 4.2 Interpolating $\mu\mathsf{LL}^\omega$ pre-proofs

Our first step to proving Theorem 2 is to extend the proof of Theorem 1 to accommodate the use of fixed-point rules $\mu$ and $\nu$, disregarding the progress condition: we will prove that cut-free pre-proofs can be *pre-proof-relevantly interpolated*. This extension however introduces new challenges due to the structure of circular pre-proofs in $\mu\mathsf{LL}^\omega$. In particular, circular pre-proofs are not well-founded. This lack of well-foundedness poses a problem for Maehara's method as mentioned above.

Another key complication is the interaction between back-edges and the splitting of sequents. Given an initial splitting of the conclusion, the ascending phase induces corresponding splittings on intermediate sequents. However, nothing ensures that the splitting assigned to each bud node will match that of its companion. This inconsistency breaks the assumptions needed to apply interpolation. Fortunately, this issue can be addressed with the following lemma (relying on a simple combinatorial argument on the number of splittings) easily adapted from Shamkanov [Sha14]:

**Lemma 1.** Let $\mathscr{P}$ be a $\mu\mathsf{LL}^\omega$ pre-proof of a sequent $\vdash \Gamma$. There exists a splitting-invariant pre-proof $\mathscr{P}'$ having the same infinite unfolding as $\mathscr{P}$, obtained by unfolding some back-edges of $\mathscr{P}$. That is, for any initial splitting $(\Gamma_l, \Gamma_r)$, the ascending phase yields a decorated derivation from $\mathscr{P}'$ such that for every bud $B$, its companion $C$ has the same splitting.

Considering a splitting-invariant circular pre-proof, it is possible to formulate an interpolation proof, neglecting validity. We can do this by maintaining the previously explained ascending phase and adding the cases for the bud and companion nodes in the descending phase, viewing buds as special axioms rules:

- **Bud nodes:** To each bud node $B$ we associate the rule $\dfrac{\vdash \Gamma_l, X_B \quad \vdash \Gamma_r, X_B}{\vdash \Gamma_l, \Gamma_r}\,(\mathsf{Cut})$ with splitting $(\Gamma_l, \Gamma_r)$ to initiate the cut introduction (ensuring that variable $X_B$ is used uniquely – recall that $X_B^\perp = X_B$).
- **Companion nodes:** Suppose we reach a companion node with a split sequent $\vdash \Gamma_l, \Gamma_r$ of the bud nodes associated with the variables $X_1, \dots, X_n$. (We consider multiple variables, since it could be the case that multiple bud nodes point to the same companion.) Due to the descending phase, we then have $\dfrac{\dfrac{\pi_l}{\vdash \Gamma_l, I} \quad \dfrac{\pi_r}{\vdash \Gamma_r, I^\perp}}{\vdash \Gamma_l, \Gamma_r}\,(\mathsf{Cut})$ where $\pi_l$ (resp. $\pi_r$) has some leaves $(\vdash \Gamma_l^1, X_1), \dots, (\vdash \Gamma_n^1, X_n)$ (resp. $(\vdash \Gamma_r^1, X_1), \dots, (\vdash \Gamma_r^n, X_n)$), and $I$ has $X_1, \dots, X_n$ as free variables, as well as other free variables related to other buds whose companion has not been yet reached. We modify the proof as in Figure 1, where $p$ is a permutation over $\{1, \dots, n\}$, and each $\sigma_i \in \{\mu, \nu\}$ (with $(\mu)^\perp = \nu$). The choice of the

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\pi_l[I_1/X_{p(1)}]\ldots[I_n/X_{p(n)}]}{\vdash \Gamma_l, I[I_1/X_{p(1)}]\ldots[I_n/X_{p(n)}]}}{\vdots}\,(\sigma_n)}{\vdash \Gamma_l, \sigma_2 X_{p(2)}\ldots\sigma_n X_{p(n)} I[I_1/X_{p(1)}] = I_2}\,(\sigma_2)}{\vdash \Gamma_l, \sigma_1 X_{p(1)}\ldots\sigma_n X_{p(n)} I = I_1}\,(\sigma_1) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\pi_r[I_1^{\perp}/X_{p(1)}]\ldots[I_n^{\perp}/X_{p(n)}]}{\vdash \Gamma_r, I[I_1^{\perp}/X_{p(1)}]\ldots[I_n^{\perp}/X_{p(n)}]}}{\vdots}\,(\sigma_n^{\perp})}{\vdash \Gamma_r, \sigma_2^{\perp} X_{p(2)}\ldots\sigma_n^{\perp} X_{p(n)} I[I_1^{\perp}/X_{p(1)}] = I_2^{\perp}}\,(\sigma_2^{\perp})}{\vdash \Gamma_r, \sigma_1^{\perp} X_{p(1)}\ldots\sigma_n^{\perp} X_{p(n)} I^{\perp} = I_1^{\perp}}\,(\sigma_1^{\perp})}{\vdash \Gamma_l, \Gamma_r}\,(\mathsf{Cut})$$

Figure 1: Companion node's step in the pre-proof interpolation process

$\sigma_i$ and the permutation $p$ is arbitrary since we neglect validity of the proof for now.

The previous steps updated each leaf $\vdash \Gamma_l^i, X_i$ (resp. $\vdash \Gamma_r^i, X_i$) of $\pi_l$ (resp. $\pi_r$) to $\vdash \Gamma_l^i, I_i$ (resp. $\vdash \Gamma_r^i, I_i^{\perp}$). Thus, it is possible to map each bud with variable $X_i$ to the sequent where $I_k$ appears, with $p(k) = i$. In the end, when the descending phase reaches the root of the proof, we get a triple $(I, \pi_l, \pi_r)$ such that (i) $I$ is a $\mu\mathsf{LL}^{\omega}$ formula since all free variables have been bound, (ii) $\pi_l$ and $\pi_r$ are circular pre-proofs with conclusions $\vdash \Gamma_l, I$ and $\vdash \Gamma_r, I^{\perp}$ respectively, (iii) $I$ is in the common language of $\Gamma$ and $\Delta$, by the descending phase from the proof of Theorem 2. Cutting $\pi_l$ and $\pi_r$ results in a finite representation of a proof of which the cuts can be eliminated reaching $\pi$ as a limit: since the interpolant is synthesized by cut-introduction, their cut-elimination progressively reconstructs the infinite unfolding of $\pi$.

### 4.3 Interpolating strongly valid proofs

A key observation in the interpolation process is that the initial splitting of formulas in the conclusion determines how threads are distributed between the two sides of the interpolated pre-proof. Specifically, if a thread $\tau$ passes through a formula $F$ in the conclusion of the original pre-proof, and $F \in \Gamma_l$ (resp. $F \in \Gamma_r$), then $\tau$ will be assigned to the left (resp. right) side of the interpolated pre-proof. Since the interpolation process duplicates the structure of the original pre-proof on both sides, every infinite path $(v_i)$ is also duplicated, yielding two copies: $(v_i)_l$ and $(v_i)_r$. If $\tau$ validates $(v_i)$ in the original proof and is assigned to, say, the left side, then the corresponding path $(v_i)_r$ on the right side requires a new thread for validation. This is where the interpolant formula $I$ plays a crucial role: it acts as a bridge, enabling the validation of such "unthreaded" paths on the opposite side.

This part being still in progress, we very briefly outline it here. The results rely on the structural conditions of *tree-compatibility*, by Sprenger and Dam [SD03], and *induction orders*, as well as on adapting Brotherston's *trace manifolds* [Bro06] to our setting by defining a corresponding structure called *strongly valid thread manifolds*. From this structure, we can extract an induction order and define a refined version, referred to as *strongly valid ordered thread manifolds* (the definitions would be discussed in the talk). This process is summarised in the following two lemmas:

**Lemma 2.** Any $\mu\mathsf{LL}^{\omega}$ pre-proof is strongly valid iff it has a strongly valid thread manifold.

**Lemma 3.** A $\mu\mathsf{LL}^{\omega}$ pre-proof $\mathscr{P}$ has a strongly valid thread manifold iff there exists an induction order $\lhd$ for $\mathscr{P}$ and $\mathscr{P}$ has a strongly valid ordered thread manifold with respect to $\lhd$.

We can now state our conjecture regarding tree-compatibility and its role in enabling interpolation:

**Conjecture 1.** Let $\mathscr{P}$ be a cut-free, splitting-invariant, and strongly valid $\mu\mathsf{LL}^{\omega}$ pre-proof. If $\mathscr{P}$ is tree-compatible, then it admits a strong validity-preserving interpolation.

## References


[AL19] Bahareh Afshari and Graham E. Leigh. Lyndon interpolation for modal $\mu$-calculus. In Aybüke Özgün and Yulia Zinova, editors, *TbiLLC 2019*, volume 13206 of *LNCS*, pages 197–213. Springer, 2019.

[Bro06] James Brotherston. Sequent calculus proof systems for inductive definitions. 2006.

[Cra57] William Craig. Three uses of the herbrand-gentzen theorem in relating model theory and proof theory. *J. Symb. Log.*, 22(3):269–285, 1957.

[Dou17] Amina Doumane. *On the infinitary proof theory of logics with fixed points*. Theses, Université Sorbonne Paris Cité, June 2017.

[Mae60] Shoji Maehara. On the interpolation theorem of Craig. *Sūgaku*, 12:235–237, 1960.

[Sau25] Alexis Saurin. Interpolation as cut-introduction. In Maribel Fernandez, editor, *FSCD*. LIPICS, 2025.

[SD03] Christoph Sprenger and Mads Dam. On the structure of inductive reasoning: Circular and tree-shaped proofs in the $\mu$-calculus. In A. Gordon, editor, *FOSSACS*, pages 425–440. Springer, 2003.

[Sha14] Daniyar S. Shamkanov. Circular proofs for Gödel-Löb logic. *Mathematical Notes*, 96(3):575–585, 2014.