

Compression for Coinductive Infinitary Rewriting: A Generic Approach, with Applications to Cut-Elimination for Non-Wellfounded Proofs

Short abstract*

Rémy Cerda

Université Paris Cité, CNRS, IRIF,
F-75013, Paris, France
Università di Bologna, Italy
Remy.Cerda@math.cnrs.fr

Alexis Saurin

Université Paris Cité, CNRS, IRIF,
F-75013, Paris, France
INRIA π^3 , Paris, France
Alexis.Saurin@irif.fr

The “traditional” line of work on infinitary rewriting is based on ordinal-indexed strongly convergent rewriting sequences (*i.e.* rewriting sequences that do not only converge in the topological sense, but such that in addition the computation steps occur deeper and deeper in the rewritten objects). A key property of such rewriting systems is *compression*, that is, the fact that rewriting sequences of arbitrary ordinal length can be compressed to sequences of length ω . For example consider the first-order rewriting rules $a \rightarrow_1 f(g(a))$ and $g(f(x)) \rightarrow_2 f(x)$, then the (strongly converging) rewriting sequence

$$a \rightarrow_1 f(g(a)) \rightarrow_1^\infty f(g(f(g(\dots)))) \rightarrow_2 f(f(g(\dots))) \rightarrow_2^\infty f(f(f(\dots)))$$

is of ordinal length $\omega \cdot 2$ but can be compressed by interleaving the rewriting steps:

$$a \rightarrow_1 f(g(a)) \rightarrow_1 f(g(f(g(a)))) \rightarrow_2 f(f(g(a))) \rightarrow_{1,2}^\infty f(f(f(\dots))).$$

This example illustrates the key benefit of compression: in a strongly converging rewriting sequence of length ω , finite approximations of the limit are computed in finite time (whereas in the example it takes $\omega + 1$ steps to produce the two outermost f). This “approximation” or “continuity” property enjoyed by rewriting sequences of length ω is at the heart of most practical motivations for the use of infinitary rewriting. For example, in the infinitary λ -calculus, it is the reason why infinitary rewriting allows one to provide an easy proof of the continuous approximation theorem [5], a result that is of paramount importance for the classical study of the λ -calculus. To the best of our knowledge, compression lemmas have been proved for three kinds of term rewriting systems: left-linear first-order rewriting [11, 12, 15, 8], infinitary λ -calculi [10, 1], and fully-extended, left-linear higher-order rewriting [13].

As an alternative to the traditional definition of infinite objects *via* ideal or metric completion, a more recent and very fruitful line of work is based on the use of *coinduction*: the metric completion of any algebraic type (*e.g.* a type of terms) can indeed be described as the corresponding coalgebraic type [2]. This coinductive reformulation has been in particular extensively conducted in the setting of infinitary λ -calculi [9, 7, 6, 5]. However, a generic presentation of coinductive infinitary rewriting was designed only recently by Endrullis, Hansen, Hendriks, Polonsky and Silva [8], even though solely for first-order rewriting. In the present work, we propose a framework for describing arbitrary non-wellfounded objects and rewriting systems acting on such objects, we extend the correspondence between topology-based

*A longer version of this work is available at <https://arxiv.org/abs/2510.08420>.

and coinduction-based rewriting to this generic framework (section 1), and we provide a coinductive characterisation of the compression property (section 2).

We also showcase one interesting application, namely compression of infinitary cut-elimination for the non-wellfounded proof system μMALL^∞ for multiplicative-additive linear logic with fixed points (section 3). This compression result is particularly useful as it allows to prove cut-elimination of various other systems (including the similar one for full linear logic, μLL^∞) [16, 3]; and whereas μMALL^∞ did not fit the usual presentation of infinitary rewriting, our framework is generic enough to encompass it.

These achievements unify and complete several threads of the literature and are meant to provide a reasonable level of generality for future investigations into coinductive infinitary rewriting. This is also a first step towards developing non-wellfounded proof theory (especially of the system μLL^∞ for linear logic with fixed points) in a coinductive setting, which is our longer term goal: a result we typically aim at is a fully coinductive proof of cut-elimination for μLL^∞ .

1 Infinitary rewriting of arbitrary non-wellfounded derivations

We introduce a description of rewriting for arbitrary non-wellfounded derivations. To build the latter, we first fix a set \mathcal{S} of *statements*. It is typically an inductively defined set using the singleton type, one or more alphabets, as well as tuples, lists, multisets, etc. For instance one could consider the set of two-sided sequents in a given logic, encoded as the set of pairs of lists of formulæ.

Definition 1. A derivation rule (r) is given by (i) its arity $\text{ar}(r) \in \mathbf{N}$, (ii) a partial function $r : \mathcal{S}^{\text{ar}(r)} \rightarrow \mathcal{S}$ mapping its premisses to its conclusion, (iii) a map $\text{coind}_r : [1, \text{ar}(r)] \rightarrow \{0, 1\}$ indicating the (co)inductive behaviour of each premiss. It is represented as follows:

$$\frac{\underbrace{S_1} \quad \cdots \quad \underbrace{S_{\text{ar}(r)}}}{r(S_1, \dots, S_{\text{ar}(r)})} (r) \quad (1)$$

where the dashed line under S_i represents a full line whenever $\text{coind}_r(i) = 1$, and an absence of line otherwise. (When we apply such a rule r to some arguments we implicitly suppose that they belong to its domain of definition.) We denote by $\text{DT}_{\mathcal{D}}^\infty$ the set of all (non-wellfounded) derivation trees generated by a family \mathcal{D} of derivation rules such that all infinite branches cross infinitely many double lines (i.e. coinductive premisses).

Example 2. 1. Pre-proofs in your favourite non-wellfounded proof system, e.g. for some logics with fixed points, can be presented as such a set $\text{DT}_{\mathcal{D}}^\infty$.

2. The set T_Σ^∞ of infinitary first-order terms [11] on the signature Σ , arity map ar and variables \mathcal{V} , can be seen as the derivation trees generated by $\mathcal{S} := \{\bullet\}$ and $\mathcal{D}_\Sigma := \{\text{Var}_x : \mathcal{S}^0 \rightarrow \mathcal{S} \mid x \in \mathcal{V}\} \cup \{\text{Cons}_c : \mathcal{S}^{\text{ar}(c)} \rightarrow \mathcal{S} \mid c \in \Sigma\}$, together with $\text{coind}_{\text{Cons}_c}(i) := 1$ for all c and i (i.e. all constructors are translated into derivations rules whose premisses are all underlined).

3. Similarly, abc-infinitary λ -terms [10] can be seen as the derivation trees generated by $\mathcal{S} := \{\bullet\}$ and $\mathcal{D}_\lambda^{abc} := \{\text{Var}_x : \mathcal{S}^0 \rightarrow \mathcal{S} \mid x \in \mathcal{V}\} \cup \{\text{Abs}_x : \mathcal{S} \rightarrow \mathcal{S} \mid x \in \mathcal{V}\} \cup \{\text{App} : \mathcal{S}^2 \rightarrow \mathcal{S}\}$, together with $\text{coind}_{\text{Abs}_x}(1) := a$, $\text{coind}_{\text{App}}(1) := b$ and $\text{coind}_{\text{App}}(2) := c$ (i.e. the booleans a , b and c describe whether respectively λ -abstraction, left side and right side of application act inductively or coinductively).

By abuse of notation, we write $s = r(s_1, \dots, s_k)$ to express that (i) the last rule of s is (r) , so that there are statements $S_1, \dots, S_k \in \mathcal{S}$ such that s has conclusion $r(S_1, \dots, S_k)$, (ii) each derivation s_i has

conclusion S_i and is the subtree rooted at the i th premiss of the concluding (r) . Using this notation, a more formal definition of the set $\text{DT}_{\mathcal{D}}^{\infty}$ can be given by the following coalgebra in the category of sets:

$$\text{DT}_{\mathcal{D}}^{\infty} := \nu X_1 \cdot \mu X_0 \cdot \prod_{r \in \mathcal{D}} r \left(X_{\text{coind}_r(1)}, \dots, X_{\text{coind}_r(\text{ar}(r))} \right).$$

This construction can also be performed in the category of nominal sets for handling α -equivalence of terms [14, 4].

Definition 3. A set $\longrightarrow_0 \subseteq \text{DT}_{\mathcal{D}}^{\infty} \times \text{DT}_{\mathcal{D}}^{\infty}$ of zero steps generates a relation \longrightarrow by the following set of inductive rules:

$$\frac{s_i \longrightarrow_d s'_i \quad 1 \leq i \leq \text{ar}(r)}{r(s_1, \dots, s_i, \dots, s_{\text{ar}(r)}) \longrightarrow_{d+\text{coind}_r(i)} r(s_1, \dots, s'_i, \dots, s_{\text{ar}(r)})}$$

where (only for the next few lines) we annotate \longrightarrow with the depth at which the rewriting step occurs.

The traditional way of defining infinitary rewriting [11] goes as follows: the set $\text{DT}_{\mathcal{D}}^{\infty}$ is equipped with a metric \mathbf{d} defined by $\mathbf{d}(s, t) := \inf \{2^{-d} \mid \lfloor s \rfloor_d = \lfloor t \rfloor_d\}$ where $\lfloor s \rfloor_d$ is the object obtained by pruning s above d coinductive rules (*i.e.* double lines). Then given an ordinal γ , a *rewriting sequence* of length γ from s_0 to s_γ is the data of derivation trees $(s_\delta)_{\delta < \gamma}$ together with rewriting steps $(s_\delta \longrightarrow_{d_\delta} s_{\delta+1})_{\delta < \gamma}$. It is *strongly converging* if for all limit ordinal $\gamma' \leq \gamma$, (i) $\lim_{\delta \rightarrow \gamma'} s_\delta = s_{\gamma'}$ and (ii) $\lim_{\delta \rightarrow \gamma'} d_\delta = \infty$. We write $s_0 \longrightarrow^\infty s_\gamma$ whenever there is a strongly converging rewriting sequence from s_0 to s_γ .

The following result extends (and slightly adapts) [8], where a similar characterisation is proved for first-order rewriting.

Theorem 4. \longrightarrow^∞ is the union of all relations $\longrightarrow_\gamma^\infty$ (for ordinals $\gamma < \omega_1$) defined by the rule (split), which is defined using auxiliary relations $\longrightarrow_\gamma^\infty$ defined by rules (lift_r) (one for each $r \in \mathcal{D}$), as follows:

$$\frac{s \overset{\gamma, m}{\rightsquigarrow} s' \quad s' \longrightarrow_\gamma^\infty t}{s \longrightarrow_\gamma^\infty t} \text{ (split)} \quad \frac{s_1 \overset{\gamma}{\rightsquigarrow} s'_1 \quad \dots \quad s_{\text{ar}(r)} \overset{\gamma}{\rightsquigarrow} s'_{\text{ar}(r)}}{r(s_1, \dots, s_{\text{ar}(r)}) \longrightarrow_\gamma^\infty r(s'_1, \dots, s'_{\text{ar}(r)})} \text{ (lift}_r\text{)}$$

where the i th premiss of (lift_r) is coinductive iff $\text{coind}_r(i) = 1$, and $s \overset{\gamma, m}{\rightsquigarrow} s'$ denotes any sequence $s \longrightarrow^* s'_1 \xrightarrow[\delta_1]{\infty} t_1 \longrightarrow^* s'_2 \xrightarrow[\delta_2]{\infty} \dots \xrightarrow[\delta_m]{\infty} t_m \longrightarrow^* s'$ such that $\forall 1 \leq i \leq m, \delta_i < \gamma$.

2 A generic compression lemma

As said in the introduction of the paper, an infinitary rewriting system (presented by topological means) is “compressible” when any rewriting sequence can be turned into a sequence of length at most ω with same source and target. We want to express the same property in our coinductive setting, hence a first step is to translate coinductively what a “rewriting sequence of length at most ω ” is.

Lemma 5. For $s, t \in \text{DT}_{\mathcal{D}}^{\infty}$, $s \longrightarrow_0^\infty t$ iff there is a strongly converging sequence of length at most ω from s to t .

As a consequence, we say that \longrightarrow has the *compression property* whenever $\longrightarrow^\infty = \longrightarrow_0^\infty$. We are now able to give a characterisation of the property in our coinductive setting:

Theorem 6. We define the following properties of the rewriting relation \longrightarrow :

$$\begin{aligned} \mathfrak{P}_\delta &: \forall n \in \mathbf{N}, \forall s, s' \in \text{DT}_{\mathcal{D}}^\infty, s \xrightarrow[\delta, n]{\rightsquigarrow} s' \Rightarrow \exists s'' \in \text{DT}_{\mathcal{D}}^\infty, \exists \varepsilon < \delta, s \longrightarrow^* s'' \xrightarrow{\varepsilon}^{\infty*} s', \\ \Omega &: \forall \delta, \forall s, t, t' \in \text{DT}_{\mathcal{D}}^\infty, \mathfrak{P}_\delta \wedge s \xrightarrow{\delta}^\infty t \longrightarrow t' \Rightarrow \exists s' \in \text{DT}_{\mathcal{D}}^\infty, s \longrightarrow^* s' \xrightarrow{\delta}^\infty t'. \end{aligned}$$

Then \longrightarrow has the compression property iff the property Ω holds.

The property Ω seems quite convoluted at first sight, but is in fact not very surprising: it essentially means that given a certain induction hypothesis (namely \mathfrak{P}_δ), a rewriting sequence of ordinal length $\delta + 1$ (for an arbitrary infinite ordinal δ) can be turned into an equivalent sequence of length $p + \delta = \delta$ (for some $p \in \mathbf{N}$). When one tries to prove a compression lemma in a traditional way, using a transfinite induction and topological arguments, the key case of the proof is exactly the same, namely the case of a successor ordinal [17, § 12.7]. Therefore, this theorem can be seen as a “factorisation” of the compression property: the property Ω is the part of the proof that remains to be adapted to each rewriting system.

In the long version of this work we detail the application of theorem 6 to left-linear first-order rewriting (as presented in example 2), which is the standard example of a compression lemma. Here we focus on another compression result, proved topologically in [16], and show how it fits into our formalism.

3 Compressing μMALL^∞ cut-elimination sequences

We now prove that cut-elimination for the non-wellfounded proof system μMALL^∞ (for multiplicative-additive linear logic with fixed points) is compressible. For the definition of this system and its encoding in the formalism introduced in section 1, we refer to the long version of this paper; in short, we define the set \mathcal{S} of statements to be the set of (one-sided) μMALL sequents, and $\text{DT}_{\mu\text{MALL}^\infty}^\infty$ to be the set of all derivations trees generated from \mathcal{S} by the rules of the system, including the *multicut* rule which is an n -ary cut merging a finite tree of cuts. The derivation trees correspond to μMALL^∞ pre-proofs.

Cut-elimination theorems for non-wellfounded proof systems rely on a rewriting relation “moving the cuts upwards” so that the corresponding infinitary rewriting produces a cut-free derivation. For μMALL^∞ this rewriting relation is defined by three kinds root rewriting steps: (i) steps handling technicalities related to multicuts, *e.g.* merging a cut into a multicut, (ii) *principal steps*, corresponding to the situation where dual formulæ are (multi)cut against each other, for example:

$$\frac{\frac{\frac{\bar{Z}}{\vdash \Gamma, F_0} \quad \frac{\bar{Z}}{\vdash \Gamma, F_1}}{\vdash \Gamma, F_0 \& F_1} (\&) \quad \frac{\bar{Z}}{\vdash \Delta, F_i^\perp} (\oplus_{i, F_{i-1}})}{\vdash \Delta, F_0^\perp \oplus F_1^\perp} (\text{mcut}_{k+2, \bar{n}, \perp})}{\vdash H} \longrightarrow \frac{\bar{Z}}{\vdash \Gamma, F_i} \quad \frac{\bar{Z}}{\vdash \Delta, F_i^\perp}}{\vdash H} (\text{mcut}_{k+2, \bar{n}, \perp})$$

(iii) *commutative steps*, corresponding to the situation where a multicut is permuted with the last rule of one of its premisses, for example:

$$\frac{\frac{\bar{Z}}{\vdash \Gamma, F \wp G} (\wp) \quad \frac{\bar{Z}}{\vdash \Gamma, F \wp G} (\wp)}{\vdash H, F \wp G} (\text{mcut}_{k+1, \bar{n}, \perp}) \longrightarrow \frac{\bar{Z}}{\vdash \Gamma, F, G} (\text{mcut}_{k+1, \bar{n}', \perp})}{\vdash H, F, G} (\wp) \quad \frac{\bar{Z}}{\vdash H, F \wp G} (\wp)}$$

Cut-elimination is the relation \longrightarrow on $\text{DT}_{\mu\text{MALL}^\infty}^\infty$ generated *via* definition 3 from these zero steps.

Theorem 7 (compression). \longrightarrow satisfies the property Ω .

Proof. We define a *derivation prefix* $p(*_1, \dots, *_k)$ to be a finite tree inductively built from the rules of μMALL^∞ , with pairwise distinct leaves $*_1, \dots, *_k$ (which are fresh symbols). We denote by $p(s_1, \dots, s_k)$ the derivation tree obtained by substituting a derivation tree s_i to each symbol $*_i$. By induction on arbitrary cut-elimination steps $s \longrightarrow t$ (as defined in definition 3), one can show that any such step has the shape $p(s_1, \dots, s_k) \longrightarrow q(s_1, \dots, s_k)$ for some derivation prefixes p and q . Indeed is true for all cut-elimination root steps and the induction case is trivial. As a consequence, if we take an ordinal δ and $s, t, t' \in \text{DT}_{\mu\text{MALL}^\infty}^\infty$ such that \mathfrak{P}_δ and $s \xrightarrow{\delta}^\infty t \longrightarrow t'$, we can write $t = p(t_1, \dots, t_n)$ and $t' = q(t_1, \dots, t_n)$.

Then we shall prove two lemmas that appear in all proofs of compression:

- *Pattern extraction.* If \mathfrak{P}_δ holds and $s \xrightarrow{\delta}^\infty p(t_1, \dots, t_k)$, then there exist $s'_1, \dots, s'_k \in \text{DT}_{\mu\text{MALL}^\infty}^\infty$ such that $s \longrightarrow^* p(s'_1, \dots, s'_k)$ and for all $i \in [1, k]$, $s'_i \xrightarrow{\delta}^\infty t_i$. The proof is by a straightforward induction over p , using \mathfrak{P}_δ and its following consequence: if $s \xrightarrow{\delta}^\infty t$ then there exists $s' \in \text{DT}_{\mathcal{D}}^\infty$ such that $s \longrightarrow^* s' \xrightarrow{\delta}^\infty t$.
- *Pattern filling.* If for all $i \in [1, k]$, $s'_i \xrightarrow{\delta}^\infty t_i$ then $q(s'_1, \dots, s'_k) \xrightarrow{\delta}^\infty q(t_1, \dots, t_k)$. The proof is by induction on q .

Using these lemmas we can conclude: $s \longrightarrow^* p(s'_1, \dots, s'_k) \xrightarrow{\delta}^\infty q(t_1, \dots, t_n) = t'$. \square

References

- [1] P. Bahr. “Partial Order Infinitary Term Rewriting and Böhm Trees”. In: *Proceedings of the 21st International Conference on Rewriting Techniques and Application*. 2010. DOI: 10.4230/LIPICS.RTA.2010.67.
- [2] M. Barr. “Terminal coalgebras in well-founded set theory”. In: *Theoretical Computer Science* 114.2 (1993). DOI: 10.1016/0304-3975(93)90076-6.
- [3] E. Bauer and A. Saurin. *A uniform cut-elimination theorem for linear logics with fixed points and super exponentials*. To appear in the proceedings of CSL 2026. URL: <https://arxiv.org/abs/2506.14327>.
- [4] R. Cerda. “Nominal Algebraic-Coalgebraic Data Types, with Applications to Infinitary λ -Calculi”. In: *Proceedings Twelfth Workshop on Fixed Points in Computer Science (FICS 2024)*. Ed. by A. Saurin. EPTCS 435. 2025. DOI: 10.4204/EPTCS.435.5.
- [5] R. Cerda. “Taylor Approximation and Infinitary λ -Calculi”. Theses. Aix-Marseille Université, 2024. URL: <https://hal.science/tel-04664728>.
- [6] Ł. Czajka. “A new coinductive confluence proof for infinitary lambda calculus”. In: *Logical Methods in Computer Science* 16.1 (2020). DOI: 10.23638/LMCS-16(1:31)2020.
- [7] J. Endrullis and A. Polonsky. “Infinitary Rewriting Coinductively”. In: *18th International Workshop on Types for Proofs and Programs (TYPES 2011)*. 2013. DOI: 10.4230/LIPICS.TYPES.2011.16.
- [8] J. Endrullis et al. “Coinductive Foundations of Infinitary Rewriting and Infinitary Equational Logic”. In: *Logical Methods in Computer Science* 14.1 (2018). DOI: 10.23638/LMCS-14(1:3)2018.
- [9] F. Joachimski. “Confluence of the coinductive λ -calculus”. In: *Theoretical Computer Science* 311.1-3 (2004). DOI: 10.1016/S0304-3975(03)00324-4.
- [10] R. Kennaway, J. W. Klop, R. Sleep, and F.-J. de Vries. “Infinitary lambda calculus”. In: *Theoretical Computer Science* 175.1 (1997). DOI: 10.1016/S0304-3975(96)00171-5.
- [11] R. Kennaway, J. W. Klop, R. Sleep, and F.-J. de Vries. “Transfinite Reductions in Orthogonal Term Rewriting Systems”. In: *Information and Computation* 119.1 (1995). DOI: 10.1006/inco.1995.1075.
- [12] J. Ketema. “Reinterpreting Compression in Infinitary Rewriting”. en. In: *23rd International Conference on Rewriting Techniques and Applications (RTA 2012)*. 2012. DOI: 10.4230/LIPICS.RTA.2012.209.
- [13] J. Ketema and J. G. Simonsen. “Infinitary Combinatory Reduction Systems”. In: *Information and Computation* 209.6 (2011). DOI: 10.1016/j.ic.2011.01.007.

- [14] A. Kurz, D. Petrişan, P. Severi, and F.-J. de Vries. “Nominal Coalgebraic Data Types with Applications to Lambda Calculus”. In: *Logical Methods in Computer Science* 9.4 (2013). DOI: 10.2168/lmcs-9(4:20)2013.
- [15] C. Lombardi, A. Ríos, and R. de Vrijer. “Proof Terms for Infinitary Rewriting”. In: *Rewriting and Typed Lambda Calculi (RTA 2014, TLCA 2014)*. 2014. DOI: 10.1007/978-3-319-08918-8_21.
- [16] A. Saurin. “A Linear Perspective on Cut-Elimination for Non-wellfounded Sequent Calculi with Least and Greatest Fixed-Points”. In: *Automated Reasoning with Analytic Tableaux and Related Methods (TABLEAUX 2023)*. 2023. DOI: 10.1007/978-3-031-43513-3_12.
- [17] Terese. *Term Rewriting Systems*. Cambridge University Press, 2003.